TABLE I.

| Institutions. | Date of Organizing Summer Courses. | Number of Weeks in the Summer Sessions. | Attendance for past five years: 1906. | 1907. | 1908. | 1909. | 1910. |
|---|---|---|---|---|---|---|---|
| California | 1891 | 6 | 706 | 522 | 661 | 819 | 1051 |
| Chicago | 1894 | 12 | 2688 | 2605 | 3050 | 3264 | 3370 |
| Columbia | 1900 | 6 | 1043 | 1392 | 1998 | 1946 | 2629 |
| Cornell | 1892 | 6 | 642 | 755 | 841 | 889 | 987 |
| Harvard | 1871 | 6 | 779 | 809 | 1116 | 903 | 873 |
| Illinois | 1894 | 9 | 423 | 502 | 555 | 631 | 665 |
| Indiana | 1890 | 12 | 682 | 721 | 1005 | 1139 | 1107 |
| Iowa | 1900 | 6 | 281 | 344 | 363 | 363 | 347 |
| Kansas | 1903 | 6 | 264 | 289 | 377 | 374 | 390 |
| Michigan | 1894 | 8 | 1034 | 1070 | 1085 | 1224 | 1234 |
| Minnesota | 1891 | 6 | 1019 | 1035 | 1218 | 1286 | 1188 |
| Missouri | 1894 | 8 | 403 | 542 | 508 | 552 | 576 |
| Oberlin | 1899 | 8 | 142 | 136 | 142 | 160 | 154 |
| Ohio State | 1905 | | 385 | 425 | 508 | 642 | 639 |
| Pennsylvania | 1904 | 6 | 505 | 707 | 879 | 826 | 1219 |
| Texas | 1899 | 7 | 444 | 580 | 625 | 741 | 848 |
| Washington State | 1904 | 6 | 198 | 243 | 235 | 288 | 303 |
| Wisconsin | 1899 | 6 | 650 | 750 | 1025 | 1125 | 1253 |

Remarks on Table 1. At the University of California, summer courses have been given since 1891, but the first regular summer session was held in 1900, since which time full credit toward the baccalaureate degree has been allowed for all summer courses of college grade. This is typical of many other institutions, for instance, at the University of Illinois, summer courses were given since 1894, but the regular sessions granting credit for work of a college grade began in 1900.

The figures for the University of Minnesota and for the University of Texas include those registered for elementary work as well as those taking work of college grade. At the University of Texas the summer session is combined with a summer normal school. This arrangement prevails also in some other cases, for instance, at Tulane University, where the summer session was introduced in 1910.

MATHEMATICS COURSES IN THE SUMMER SESSIONS.

The foregoing general statement with reference to the development of the summer sessions in American universities forms an appropriate background for the consideration of the work of mathematics in particular in these sessions. It appears that the courses in mathematics have been regarded in most cases as of special importance in these summer curricula. In fact, in several instances the work began with courses in mathematics and perhaps no other subjects, and was gradually extended to include courses in all the regular departments as the number of students increased and the demand became apparent.

The courses offered in mathematics may be roughly divided into two

classes: (1) those intended primarily to emphasize the pedagogical aspect of the subject, and (2) those intended primarily to develop the subject matter for its own sake or as a prerequisite to other courses. The pedagogical courses include critical studies of the various elementary and secondary branches with reference to scientific interpretation and methods of presentation, and also studies in the history of mathematics with special reference to the needs of teachers in elementary and secondary schools. As yet there appear to have been no pedagogical courses offered for the special benefit of teachers of college mathematics, but it may be said that practically every course offered in the summer session becomes in a certain sense a pedagogical course, since so large a proportion of the students are teachers who are keenly alive to their opportunities for observation. In fact it is a common occurrence in these summer courses for students to register for the purpose of studying the teacher quite as much as for studying the subject. Moreover, it is believed by many that the best pedagogical training comes through careful and diligent study of the subject matter, under the guidance of an inspiring teacher who knows how to exhibit good methods and to impress them upon the students by example, rather than by precept. However, there is undoubtedly a growing tendency to attach greater importance to definite, scientific study of the art of teaching, and this tendency is shown in the increasing number of pedagogical courses offered in colleges and universities, especially in the summer sessions, and in this respect mathematics is well represented. Naturally such courses appear most strongly in those institutions in which the department of education, or the separate school of education, is most fully developed.

Of the courses intended primarily for the study of subject matter, comparatively few extending beyond elementary calculus are offered in summer sessions, except in one or two institutions, notably at Chicago and Wisconsin. This may be accounted for in two ways: (1) because in most cases the great majority of students in the summer sessions are not prepared for advanced work, and (2) because the summer school usually has no endowment and the receipts from tuition fees are not sufficient to provide the more expensive iustruction in advanced courses where the enrollment is comparparatively small. The following statement from the former director of the Harvard summer school is fairly typical of the way in which the mathematical work has developed in the summer sessions in other institutions: "Mathematical courses were first introduced into the Harvard summer school in 1891. Courses in solid geometry, plane trigonometry, and college algebra have been given regularly for the past twenty years. Courses in analytic geometry have been given regularly for the past ten years. Courses in calculus regularly for the past seven years. In 1903 a course was given in the theory of functions, and in 1909 a course in modern methods in geometry. In 1910 the following advanced courses were offered: Introduction to modern geometry, topics in the theory of functions, each with very small registration."

The following table shows certain data with respect to mathematics in summer sessions.

TABLE II.

| Institutions. | Registration in Mathematics Courses | | | | | Elementary Courses including Calculus | Advanced Courses beyond Calculus | Pedagogical Courses |
|---|---|---|---|---|---|---|---|---|
| | 1906 | 1907 | 1908 | 1909 | 1910 | | | |
| California | 41 | 38 | 63 | 60 | 108 | 4 | 1 | 1 |
| Chicago | 286 | 316 | 350 | 380 | 493 | 12 | 4 | 3 |
| Columbia | 130 | 160 | 170 | 210 | 275 | 7 | 3 | 2 |
| Cornell | Not reported | | | 369 | 375 | 8 | 2 | 2 |
| Harvard | 58 | 54 | 43 | 37 | 40 | 5 | 2 | 1 |
| Illinois | 110 | 136 | 152 | 183 | 164 | 5 | 4 | 1 |
| Indiana | 98 | 150 | 177 | 173 | 189 | 4 | 2 | — |
| Iowa | 46 | 80 | 73 | 68 | 61 | 7 | — | 2 |
| Kansas | 13 | 21 | 13 | 20 | 39 | 3 | 1 | — |
| Michigan | Average of 155 per year | | | | | 6 | 4 | 1 |
| Minnesota | Average of 100 per year | | | | | 3 | — | — |
| Missouri | 95 | 98 | 56 | 79 | 85 | 5 | 1 or 2 | 1 or 2 |
| Nebraska | 33 | 33 | 30 | 32 | 42 | 4 | — | — |
| Oberlin | Average of 35 per year | | | | | 4 | — | 1 |
| Ohio State | Not reported | | | | 113 | 4 | 1 | 1 |
| Pennsylvania | 88 | 129 | 151 | 121 | 184 | 11 | — | 1 |
| Texas | 348 | 420 | 454 | 446 | 342 | 3 | — | 1 |
| Washington State | 27 | 29 | 19 | 22 | 22 | 1 | — | 2 |
| Wisconsin | 142 | 117 | 249 | 225 | 242 | 9 | 4 | — |

Remarks on Table II. The data at hand do not make it clear in all cases whether the figures in the above table indicate the total registration or the actual number of different students, as in the case of Iowa and Kansas and Oberlin. The numbers for Texas include attendance upon mathematics courses in a summer normal school which is held in connection with the summer school of the university.

At the University of Pennsylvania no advanced course in mathematics was offered in the summer of 1910, though three or four courses had been offered up to that time. Also the elementary courses include some preparatory subjects which will not be given hereafter. (The data as to courses in all cases refer to 1910 unless otherwise specified.)

At the University of Iowa, while no advanced courses are scheduled, yet those students in the summer sessions who desire some advanced work are cared for in small groups as far as possible.

At the University of Kansas no advanced courses have hitherto been offered. Two are to be given in 1911.

The registration in mathematics at Indiana University and at the University of Chicago is given for the first six weeks of the summer sessions. The attendance is considerably diminished during the second six weeks. These are the only institutions reported in which the session consists of two terms of six weeks each.

U of M

At the University of Illinois, previous to 1910, courses in preparatory subjects were offered. The registration given in the table is exclusive of these courses.

## The Training of Teachers.

In general it may be said that a large number of those in attendance at the summer sessions are teachers spending a part or all of their vacation in study, but it does not follow that all are seeking the strictly pedagogical courses. In fact, very many are pursuing courses for degrees and are therefore filling out requirements or choosing elective work in subject matter in which they are interested. It is true, however, that in most cases where special courses of a pedagogical character are given they are well attended and fully appreciated. On the other hand, there is a wide difference of policy, among the institutions offering summer work, as to the usefulness of the pedagogical courses as compared with the content courses.

The following comments from reports of various institutions in the above list will illustrate these points with respect to the department of mathematics.

"As seen from our enrollment the interest in summer school mathematics is not very great, most teachers preferring work which does not demand so great a strain as the study of mathematics. Whatever work is done has a decidedly beneficial influence upon those who take it."

"Only one course in the pedagogy of mathematics has been offered and this has not been largely attended. It is not possible to form any estimate as to the influence of this work in general upon the preparation of teachers of mathematics."

"A course in algebra has been given from the first, arranged with special reference to broadening the point of view of the high school teacher. Graphical methods are developed, the theory of complex numbers presented and the notation of determinants explained. For the last four summers a course in the history of mathematics has been offered, in which the development of geometry and algebra is traced as carefully as may be from the earliest times. A large proportion of the students in mathematics are enrolled in one or the other of these two courses for teachers and their obvious interest in, and profit from, the work has been gratifying. It is believed that the direct influence of these summer courses upon the school teachers, by far the most numerous class of students in the session, is greater than that of any other work which we do during the academic year."

"Of the advanced courses given in 1910, two were special courses intended for the training of teachers in mathematics. In the one year of their existence these two special courses for teachers attracted many students who showed much interest in the work. It is the estimate of the department of mathematics here that the summer sessions have had good influence upon the preparation of teachers of mathematics."

"In general, the demand for the advanced courses in the summer sessions tends to increase while that for the most elementary courses is stationary or decreasing. Many courses, especially those of intermediate character on such subjects as modern geometry, projective geometry, and selected topics in algebra, are especially adapted to teachers. Some of these courses are very stimulating to them and show them how to use the literature and text books in their subjects more adequately. Not a few of the teachers in attendance in mathematics in the summer sessions drift later into the university."

"The special courses intended for the training of teachers in mathematics have been (1) a general course covering the usual high school work, and (2) conferences. It should be added, however, that great emphasis is laid on those aspects of all the courses which are specially helpful to teachers. The interest in these courses manifested by teachers and persons preparing to teach has been distinctly good. We think the summer work has had a decidedly beneficial influence on the preparation of teachers in mathematics. It seems repeatedly to have led teachers to enter the University for the purpose of completing their college courses. We have frequently heard encouraging remarks from teachers concerning the benefit they have derived from their summer work. Every thing considered, the summer session seems to have had even a greater influence on the efficiency of teaching than the regular sessions."

"No special courses in mathematics for the training of teachers are offered during the Summer, this work being given during the regular year."

"No course in mathematics is presented for the special purpose of teaching the students how to teach the subject, but frequent suggestions are given to that end. Most teachers who take them do so for the subject matter, as well as to observe how those in charge of the courses present them."

"All those in the course for teachers take a great interest in the subject and do much collateral reading in our extensive pedagogical library in mathematics. All express themselves as greatly benefitted by the course. Many in the course have had from three to ten years' experience in teaching."

"During the past ten years a course in the history and teaching of elementary mathematics has been given about five different times. Interest in these several courses intended for the training of teachers of mathematics, on the part of college students, has been very slight, registration in them being composed mainly of teachers in service. The influence of the summer courses upon the preparation of teachers of mathematics has been very helpful to the rather limited number of teachers who have attended the courses."

"The courses in mathematics, especially the one for the training of teachers, have called forth a good deal of interest and have been very valu-

able and inspiring. Even more satisfactory is the teachers' course which we carry throughout the college year, enrolling both those who have taught mathematics and those who are looking toward teaching in secondary schools. We are convinced that the teaching efficiency of these students is greatly increased by reason of their facing in this way the various problems that they will meet at the outset of their teaching experience."

It may be noted that the establishment of well organized schools of education, on the same basis as other professional schools, in some of the leading universities has had a marked influence upon the standards which the public schools are beginning to prescribe with respect to the preparation of their teachers, and wherever these professional schools for teachers hold summer sessions the attendance is large and increasing and the interest is rapidly growing. Some states are already prescribing by statute a minimum of work in education which must be fulfilled by all candidates for high school positions. One of the leading states in this respect is California, where a year of post-graduate work in education in the state university, or in a university of equally high standing, is demanded of all high school teachers. The work in the summer session is allowed to count toward fulfilling this requirement. This will explain the following extract from the California report. "The majority of the students in the summer sessions are from the high schools of the State, with an increasing number of graduates of other institutions who are preparing to teach in our high schools. We require a year of graduate work for the high school certificate, and have recently agreed to accept summer session residence at the rate of four summer sessions for one year of the regular session. We find considerable interest among our best high school teachers in the summer session courses, and believe we are able to help them to a broader view of the subjects which they have to teach. The demand which they make on us for these courses is increasing every year, and is larger than we are able to meet for advanced work, owing to the fact that the administration still insists that the summer session must pay for itself."

Another feature of growing significance is the fact that the number of men, in proportion to the women, is steadily increasing in the professional schools of education, as is not the case in the ordinary normal school. This is particularly noticeable in the summer sessions of the larger universities where the professional side of education is most largely developed. This seems to indicate that the demand for teachers, and particularly for men, with professional training is increasing, and teachers everywhere are beginning to recognize the necessity of such training in order to secure advancement in rank and salary. The first step is naturally the summer school and this explains the rapidly increasing attendance upon these sessions all over the country, and by all classes of teachers from the elementary schools to the high schools and colleges.

## Summer Sessions in Normal Schools.

Reports have been received from a limited number of representative normal schools in various parts of the country. In a general way it may be said that the normal schools in the East and in the far West do not hold summer sessions, and that those in the middle West do hold summer sessions, often extending through twelve weeks. There are important exceptions to this statement, and of course, there are many institutes and various kinds of summer schools for teachers which provide lectures and other forms of instruction through periods of from one to four weeks, but which are not here classed with those summer sessions which provide continuations of the regular work in normal schools.

*Extracts from a few replies:* "The two normal schools in New Jersey hold no summer session since their services are not needed in view of the large opportunities offered in the universities of New York."

"The only normal school in Massachusetts which makes much of summer work is that at Hyannis."

"The normal school at Danville, Conn., has held a four weeks session during July, beginning in 1908. Its object is to give to teachers of common schools who consider themselves unable to take the regular two years' course a brief summary of the common school subjects, what to teach and how to teach it."

"We have in Maryland two normal schools, neither of which holds a summer session."

"The State normal school at Los Angeles, California, has never held summer sessions. Generally one of the five normal schools in the state holds a summer session, but none will be held this year (1911)."

"The two state normal schools of Ohio, one at Oxford and one at Athens, hold summer sessions. The former began the Summer work in 1903 and has had an attendance in each of the last five years ranging from 550 to 700. The registration in mathematics classes has averaged about 170. For the session of 1911 nine courses in mathematics are offered, from arithmetic to trigonometry, and including two courses on the teaching of elementary mathematics."

"The state normal school at Kirksville, Missouri, has held four sessions of twelve weeks during each of the past eight years. The attendance during the summer term has varied during the past five years from 500 to 610. The estimated registration in mathematics courses each year has been about sixty per cent of the total attendance. Courses of both high school and college rank are offered from elementary algebra to calculus, including two courses in the teaching of mathematics. The influence of our mathematics courses has been to induce many students to return to the institution summer after summer for mathematics and other subjects. Many of our mathematics students discover, while here, their ability in mathematics, and develop a desire to be specialists in the subject."

"All of the Michigan state normal schools hold summer sessions. The one at Kalamazoo began on this plan at its organization in 1904. The attendance during the past seven years has increased from 117 in 1904 to 834 in 1910. The courses offered in mathematics below calculus are college algebra, trigonometry, elementary algebra, and geometry. A teachers' course in arithmetic and review arithmetic are intended for the training of teachers of mathematics. Regarding the influence of this work upon the preparation of teachers of mathematics, I will say that very great interest has been shown, especially on the part of those preparing for the grades; also many have taken work looking forward to teaching high school mathematics, and a number of graduates have continued work in the universities."

"All of the Wisconsin state normal schools hold summer sessions. The one at Platteville began two years ago with a session of six week's duration, and last year (1910) had an attendance of 265, of whom about 160 were registered in elementary mathematics courses. We have little opportunity as yet to judge as to the effect of these courses upon the preparation of teachers of mathematics."

All of the normal schools of Illinois hold summer sessions and at least one, at Normal, Illinois, has a session of twelve weeks. All these sessions are largely attended.

"The state normal school at Emporia, Kansas, has had summer sessions in some form since 1891. Mathematics and latin were the only subjects in which instruction was given at that time. The summer session soon developed into a regular feature of the institution and for a number of years past has been organized upon exactly the same basis with respect to employment of faculty, records, and requirements, as at any other portion of our school year. It constitutes one of the five terms of the year. In other words, we do in the summer school the work of one-half a semester,—nine full weeks of class work. The courses offered in mathematics have been the regular work of the normal school in arithmetic, algebra, and geometry of secondary grade; and in higher algebra, trigonometry, surveying, analytics, and calculus of college grade. The other normal schools of the state, one of them eight years old, the other ten, have had summer sessions on the same basis ever since they were organized. The attendance at Emporia, in 1910 was 1200, and the number of people enrolled in mathematics courses was 305. During the past five years the average attendance has been about 1000, the registration in mathematics approximately 250. We draw from all classes of teachers in the state including many who are doing high school work, and the general influence upon the teaching work of the state has been quite extensive. It is difficult to give any accurate or specific data with regard to this, but in all of the courses offered in mathematics it has been the intention to consider them as specially organized with reference to preparation for teaching."

Enough has been said to indicate the general influence of the normal

school summer sessions, especially in the middle West. It is clear that for teachers, especially in the common schools and in the elementary schools, these sessions afford opportunities otherwise out of their reach and insure to large numbers of teachers a growth in effectiveness and an enlarging interest in their work which they could obtain in no other way. Of these general advantages, the teachers of mathematics probably enjoy a larger share than any others, since the data seem to indicate that a relatively larger number of teachers are registered for mathematics courses than is the case in other departments.

Finally, one other phase of summer school work should be mentioned, namely, that which has been held at Chatauqua, New York, since 1878. While this began, and was long continued, largely in the form of popular lectures, in more recent years organized class work has been carried on in many departments of the school and college curriculum, including mathematics. The attendance is large and the interest is great. Much attention is given to arithmetic, algebra, and geometry, and no doubt the influence upon the teaching of these subjects is wide-spread and important.

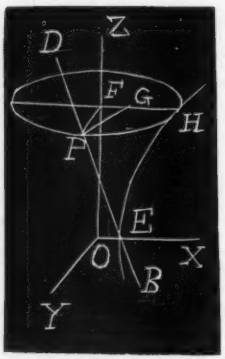
## Conclusion.

It has been seen that the summer session is a development of the last twenty years, and more extensively of the last ten years. Many colleges and most of the universities are now holding summer sessions. Particular interest centers in those institutions where schools of education have been established on a professional scale or where the department of education is recognized as of leading importance. But even in those institutions where this is not the case, much of the work of the summer session has a pedagogical character due in part to the method of presentation and in part to the critical observation of the instructor by the summer students who are themselves teachers. The normal schools, especially of the middle West, are using the summer sessions to the very great advantage of large numbers of teachers who, otherwise, would be deprived of the opportunity for advancement and growth in the profession of teaching. The department of mathematics has its full share of patronage in the summer sessions and is doing its full quota of work in this form of preparation for teaching.

# THE HYPERBOLOID AS A RULED SURFACE.*

By JAMES PERRY WILSON, Columbia University.

If a straight line is revolved around another line not in the same plane, as indicated below, it will generate a hyperboloid of one sheet.




Take the $Z$-axis as the fixed line. We can then fix the other axes so that the generating line, in its initial position $BD$, will be parallel to the $YZ$-plane, and will intersect the $X$-axis at $E$. The equations of the generating line in this position will then be of the form $x=a$, $y=mz$.

As the line is revolved around the $Z$-axis, every point on the line will generate a circle around the $Z$-axis; and the locus of the intersections of these circles with the $XZ$-plane will indicate the nature of the surface generated by the line.

Consider any point $P(a, mz, z)$ on the line $BD$. As the line is revolved around $OZ$, $P$ will describe a circle around $OZ$, whose radius $FP=\sqrt{(FG^2+GP^2)}=\sqrt{(a^2+m^2z^2)}$. This circle will intersect the $XZ$-plane at $H$; then

$$FH=x=\sqrt{(a^2+m^2z^2)}\ldots(1).$$

Since $P$ is any point on the generating line, in its initial position, equation (1) represents the locus of the intersections with the $XZ$-plane, of the circles generated by all the points on the moving line.

From (1), by rationalizing, transposing, and dividing by $a^2$, we obtain

$$\frac{x^2}{a^2}-\frac{m^2z^2}{a^2}=1.$$

Putting $\frac{a^2}{m^2}=c^2$, we obtain

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1\ldots(2),$$

which is the standard equation of the hyperbola; and the surface generated

*This note was presented by a student in a beginner's class in solid analytical geometry. Its extreme simplicity seems to warrant its publication.

by revolving $BD$ around $OZ$ is the same as that generated by revolving the hyperbola (2) around $OZ$, its conjugate axis; that is, the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=1.$$

If the moving line is perpendicular to the fixed line, it will lie wholly in the plane $Z=k$, and the hyperboloid will degenerate into the plane figure $Z=k$, $x^2+y^2 \equiv a^2$, $a$ in this case being the shortest distance between the lines $OZ$ and $BD$.

## THE SOLUTION OF AN EQUATION BY A FRAME.

By T. M. BLAKSLEE, Ames, Iowa.

The *frame* of the equation

$$f(x)=x^n+bx^{n-1}+cx^{n-2}+dx^{n-3}+\ldots=0,$$

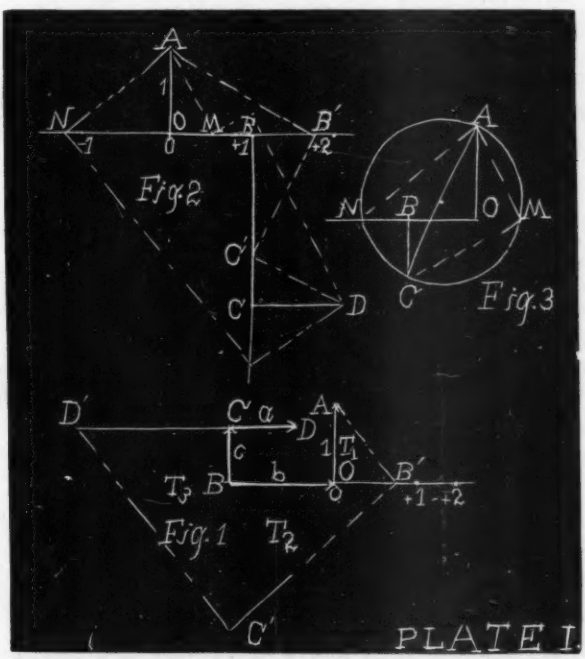



PLATE I. $OB'=r$, $C'B=r^2+br$, $D'C=r^2+br^2+c$ in Fig. 1. Also, $b'=r+b=BB'$, $c'=r^2+br+c=C'C$. In figure after 1, $b$, $c$, ... are omitted as their use is evident from Fig. 1.

(see Fig. 1) consists of a succession of strokes, $AO$, $OB$, $BC$, $CD$, ..., successive strokes being at right angles, $AO=a=-1$, $OB=-b$, $BC=c$, $CD=d$, $DE=-e$, $EF=-f$, $FG=-g$, ...

To substitute $r$ for $x$, $OB'$ representing $r$, draw the *path* $AB'$, $B'C'$, $C'D'$, ..., $T_1$, $T_2$, $T_3$, ... These are easily seen to be similar, and as $OB'$ is $r$ times $OA$, $BC$ is $r$ times $BB'$, ... Hence the values on the figure. If $f(x)=x^3+bx^2+cx+d$, $f(r)=D'D$. If $r$ is a root of $f(x)=0$, $D'$ falls at $D$.

*To draw the path* for $x=r$, place one edge of a rectangular card through $A$ and a point near $B'$, place another card against it and a third against the second and so the proper edge passes through $D$. Now $A$ and $D$ remaining fixed, slide the cards to locate $B'$ and $C'$. A little practice makes this easy. Fig. 2 solves $x^3-x^2-2x+1=0$. Horners method will give as many figures as wished. After locating one root, *e. g.*, that near $+2$, we may draw, Fig. 3, the frame of the quadratic having the other two roots. In this $b$ and $c$ are the $b'$ and $c'$ of Fig. 2. Just so in the numerical way,

$$\begin{array}{cccccccc|l} 1 & + & b & + & c & + & d & \underline{r} \\ & & r & & r^2+br & & & \\ \hline 1, & & r+b=b', & & r^2+br+c=c', & & d'=0 & \end{array}$$

$x^2+b'x+c'=0$

As the Runge Circle cuts the ray of $b$ the other two roots are real. They should check with those of Fig. 2.

As a second illustration we will take the cubic $x^3-4x^2+9x-10=0$. Fig. 4 gives one real root. As the Runge Circle (*i. e.*, the circle on the diameter $AC$, Fig. 5) does not cut the ray of $b$, the other two roots are complex.

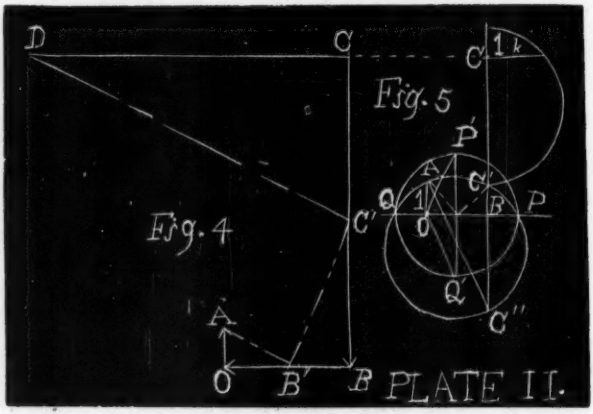



PLATE II.

PLATE II. As $OB=b$ (numerically)$=2h$, it is evident $BC'=h^3$.

Of the four methods that I have devised for the solution of a quadratic with complex roots I give two. Of these, II seems the "slicker." To consider all of them in their relations would alone be enough for a paper.

Starting with the frame $AOBC$ (Fig. 5), $b$ and $c$ being the $b'$ and $c'$ of Fig. 4, we draw the path $AB'C'$ for *equal roots*. If the roots are $h+ki$ and $h-ki$, $h=-\frac{1}{2}b$, $h^2+k^2=c$. Therefore, $C'C=c-h^2=k^2$.

I. Find $k$ as in upper part of Fig. 5. II. Make $C'C''=C'C$, and solve the frame $AOBC''$ giving the root-points $P$ and $Q$. Then $P'$ and $Q'$ are the root-points of the given quadratic frame $AOBC$, since $y^2-2hy=k^2-h^2$ gives $y=h\pm k$.

The *linkage* of $f(x)$ for a given complex value of $x$ is the succession of strokes, or links, which starting from the origin has the terminus of one link as the origin of the next, and each link represents the corresponding term of $f(x)$. If the value of $x$ used is a root of $f(x)$ the last link terminates at the origin. The point representing $x$, or "point $x$," is the root-point. The point $X$ representing $f(x)$, or "point $X$," is the function-point. As we may solve $x^n=N$ by a frame, *e. g.*, $x^3+0x^2+0x-N=0$, Fig. 6 gives an easy way of finding the lengths of $x^2$, $x^3$, $x^4$, ... Their slants being two, three, four, ... times that of $x$. Fig. 7 gives the linkages for $x^3-4x^2+9x-10=0$

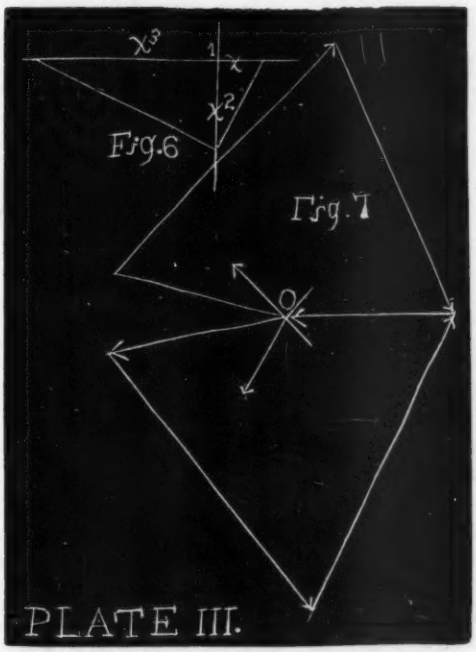



PLATE III. The successive strokes in Fig. 7 from $O$ to $O$ again are the successive terms of the polynomial $x^3-4x^2+9x-10$. Those in the upper part of the figure (light strokes) are for $x=x_2$. Those in the lower part of the figure (heavy strokes) are for $x=x_1$.

for $x_1=OP$, $x_2=OQ$, the unit being two-fifths that of Fig. 5. It is easy to show that the termini of corresponding links are conjugate points symmetrical as to the axis of reals. Hence if $X_1$ falls at the origin, so does $X_2$. We easily have a graphical proof of the theorem. Complex roots enter $f(x)=0$ in conjugate pairs.

Let us now consider a more general method in which all roots are regarded as of the form $h+ki$, where either $h$ or $k$ may be zero.

The chief interest in the method is its generality. The "double position" method used is: Having found $R_1$ and $R_2$, the function-points corresponding to $r_1$ and $r_2$, construct on $r_1r_2$ a triangle $r_1r_20$ similar to $R_1R_2O$, and take 0 as a first approximation to the root-point of the origin $O$. In Figs. 8, 9, and 10, the method is applied to $x^4-5x^3+13x^2-19x+10=0$.

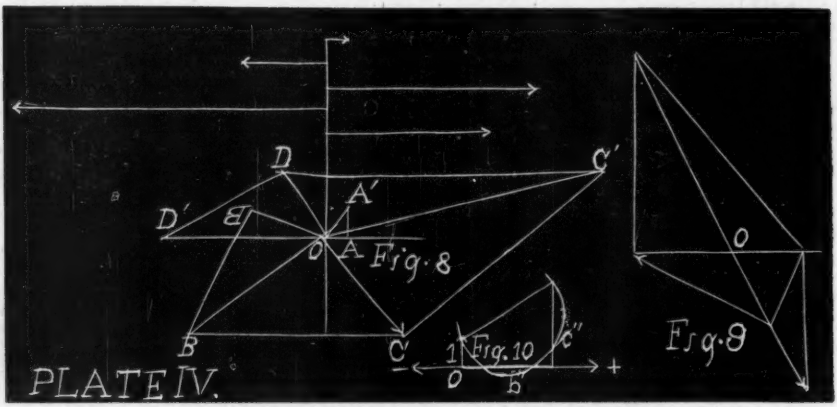



PLATE IV. Referring to Fig. 1 and the note concerning it, it is seen that, if $r$ be for $r_1$, $OA'=r$, $A'B=b$. $\therefore OB=b'$, $OB'=b'r$, $B'C=c$. $\therefore OC=c'$, $OC'=c'r$, $C'D=d$. $\therefore OD=d'$ and $OD'=d'r$. As $OD'+c=0$. Thus is the last stroke terminate at $O$, $r$ is a root.

It is to be remembered that, "Geometric multiplication consists in doing with the multiplicand (as regards slant and slide, *i. e.*, turning and stretching) what must be done to the initial unit, $(1_0)$, to obtain the multiplier;" also that, "The geometric sum of two consecutive strokes is the stroke from the first origin to the last terminus." In Fig. 8, *for the multiplication* the triangles $OAA'$, $OBB'$, $OCC'$ and $ODD'$ are similar. *For the addition*, $OB=OA'+A'B$ or $r+b=b'$, $OC=OB'+B'C=b'r+c=r^2+br+c=c'-\ldots$ In (9) $a'$, $b'$, $c'$, $d'$ of (8) are used to find $a''$, $b''$, $c''$. These are all reals. Thus in (10) we have the ordinary frame for the quadratic with roots 1 and 2. (Unit in (9) and (10) twice that in (8).) Thus the roots of $x^4-5x^3+13x^2-19x+10=0$ are $1+2i$, $1-2i$, $+1$, and $+2$.

It seems to me that, if we carry out what is touched upon in connection with Fig. 5, it will be seen that it is more logical to regard the roots as $1\pm 2i$, where $k=2$ and $\frac{3}{2}\pm(\frac{1}{2}i)i$ where $k=\frac{1}{2}i$ than to say they are $1+2i$, $1-2i$, $1+0.i$, and $2+0.i$.

# DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### ALGEBRA.

**356 Proposed by ARTEMAS MARTIN, Ph. D., Washington, D. C.**

Solve by quadratics, if possible, the equations,

$$w(x+y+z)=a, \quad x(w+y+z)=b,$$
$$y(w+x+z)=c, \quad z(w+x+y)=d.$$

[From the *Mathematical Magazine*, Vol. II, p. 256.]

No satisfactory solution of this problem has been received.

**357. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Illinois.**

Solve the system

$$\sqrt{(x^2+a^2+b^2+c^2)}=\sqrt{(y^2+b^2+c^2)}+\sqrt{(z^2+b^2+c^2)},$$
$$\sqrt{(y^2+a^2+b^2+c^2)}=\sqrt{(x^2+c^2+a^2)}+\sqrt{(z^2+c^2+a^2)},$$
$$\sqrt{(z^2+a^2+b^2+c^2)}=\sqrt{(x^2+a^2+b^2)}+\sqrt{(y^2+a^2+b^2)}.$$

**Solution by ARTEMAS MARTIN, LL.D., Editor and Publisher, Mathematical Magazine, Washington, D. C.**

Let $u=x^2+a^2+b^2+c^2$, $v=y^2+a^2+b^2+c^2$, $w=z^2+a^2+b^2+c^2$, then

$x^2+c^2+a^2=u-b^2$, $x^2+a^2+b^2=u-c^2$;

$y^2+b^2+c^2=v-a^2$, $y^2+a^2+b^2=v-c^2$;

$z^2+b^2+c^2=w-a^2$, $z^2+c^2+a^2=w-b^2$;

and $x=\sqrt{(u-a^2-b^2-c^2)}$, $y=\sqrt{(v-a^2-b^2-c^2)}$, $z=\sqrt{(w-a^2-b^2-c^2)}$.

Substituting in the given equations,

$$\sqrt{u}=\sqrt{(v-a^2)}+\sqrt{(w-a^2)}\ldots(4),$$
$$\sqrt{v}=\sqrt{(u-b^2)}+\sqrt{(w-b^2)}\ldots(5),$$
$$\sqrt{w}=\sqrt{(u-c^2)}+\sqrt{(v-c^2)}\ldots(6).$$

Squaring (4), (5), and (6), transposing, etc.,

$$2\sqrt{(v-a^2)}\sqrt{(w-a^2)}=u-v-w+2a^2\ldots(7),$$
$$2\sqrt{(u-b^2)}\sqrt{(w-b^2)}=v-u-w+2b^2\ldots(8),$$
$$2\sqrt{(u-c^2)}\sqrt{(v-c^2)}=w-u-v+2c^2\ldots(9).$$

Squaring (7), (8), and (9), transposing, etc.,

$$4vw=(u-v-w)^2+4a^2u\ldots(10),$$
$$4uw=(v-u-w)^2+4b^2v\ldots(11),$$
$$4uv=(w-u-v)^2+4c^2w\ldots(12).$$

Expanding in (10), (11), and (12), transposing, etc.,

$$2uv+2uw+2vw=u^2+v^2+w^2+4a^2u\ldots(13),$$
$$2uv+2uw+2vw=u^2+v^2+w^2+4b^2v\ldots(14),$$
$$2uv+2uw+2vw=u^2+v^2+w^2+4c^2w\ldots(15).$$

From (13), (14), and (15), we have, obviously,

$$4a^2u=4b^2v=4c^2w\ldots(16),$$

which gives $v=\frac{a^2u}{b^2}$ and $w=\frac{a^2u}{c^2}$. Substituting these values of $v$ and $w$ in (13), we get

$$u=\frac{4a^2b^4c^4}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)}.$$

Since $v=\frac{a^2u}{b^2}$ and $w=\frac{a^2u}{c^2}$, we have

$$v=\frac{4b^2a^4c^4}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)},$$

$$w=\frac{4c^2a^4b^4}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)}.$$

Substituting the above values of $u$, $v$, $w$ in the expressions for $x$, $y$, $z$, we have, finally,

$$x=\sqrt{\left(\frac{4a^2b^4c^4-2a^2b^2c^2(a^2+b^2+c^2)^2+(a^2+b^2+c^2)(a^4b^4+a^4c^4+b^4c^4)}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)}\right)},$$

$$y=\sqrt{\left(\frac{4b^2a^4c^4-2a^2b^2c^2(a^2+b^2+c^2)^2+(a^2+b^2+c^2)(a^4b^4+a^4c^4+b^4c^4)}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)}\right)},$$

$$z=\sqrt{\left(\frac{4c^2a^4b^4-2a^2b^2c^2(a^2+b^2+c^2)^2+(a^2+b^2+c^2)(a^4b^4+a^4c^4+b^4c^4)}{2a^2b^2c^2(a^2+b^2+c^2)-(a^4b^4+a^4c^4+b^4c^4)}\right)}.$$

## GEOMETRY.

378. Proposed by G. I. HOPKINS, A. M., Instructor in Mathematics and Astronomy, Manchester High School Manchester, N. H.

In the triangle $AED$, the lines $BE$ and $CE$ are drawn to the points $B$ and $C$ in the base of the triangle. If $AE=100$, $ED=125$, $BC=60$ and $\angle AEC=\angle BED=$a right angle, compute $AB$, $BE$, $EC$, and $CD$.

II. Solution by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

The solution by A. H. Holmes is correct in principle, but he has made some numerical mistake. The method of successive approximation is not only easier, but has the advantage that the work checks itself (except the last operation).

Using Mr. Holmes' notation, we have,

$$4\sin\theta=5\sin\psi\ldots(1);$$
$$25\left(\frac{1}{\cos\psi}-\cos\psi\right)+20\left(\frac{1}{\cos\theta}-\cos\theta\right)=12\ldots(2).$$

Let $\theta=35°$. Then from (1), $\sin\theta=.4589$ and $\psi=27°\ 19'$, $\cos\psi=.8885$. In (2), the first member has the value 13.955.

Let $\theta=33°$. From (1), $\psi=25°\ 50'$. First member of (2) $=12.3445$.

Let $\theta=32°\ 34'$, $\psi=25°\ 30'\ 20''$. First member of (2) $=12.0065$.

Let $\theta=32°\ 33'$, $\psi=25°\ 29'\ 41''$. First member of (2) $=11.999915$.

The work can be shortened by using interpolation just as is done in using logarithm tables.

With these last values of $\theta$ and $\psi$, we get,

$$AB=58.635,\quad BE=59.608,\quad EC=63.830,\quad CD=78.485.$$

## NOTES AND NEWS.

Dr. E. B. Stouffer, who received his Ph.D. degree from the University of Illinois in June, has been appointed instructor in mathematics in the same institution. M.

Miss Josephine Burns, who was a graduate student in the University of Illinois during the past year, has accepted a fellowship in mathematics in the University of Wisconsin for the coming year. M.

Dr. Thomas Buck and Dr. L. I. Neikirk resigned their instructorships in the University of Illinois to accept similar positions in the University of California and the University of Washington, respectively. M.

At the University of Illinois, Professor H. L. Rietz was promoted from an assisstant professorship to an associate professorship in mathematics. Professor Rietz gives courses on the theory of statistics and the mathematics of investment as well as courses in pure mathematics. M.

The fifth International Congress of Mathematicians will be held in Cambridge, England, from August 22 to August 28, 1912. It will be organized by the Cambridge Philosophical Society with the coöperation of the London Mathematical Society. This will be the first international congress of mathematicians to be held in an English-speaking country, and it is expected that a large number of Americans will attend. The preceding four congresses were held as follows: Zurich, Switzerland, August, 1897; Paris, France, August, 1900; Heidelberg, Germany, August, 1904; Rome, Italy, April, 1908. M.

B. G. Teubner of Leipzig announces that they have in preparation five large volumes on the teaching of mathematics in Germany. A number of parts have already appeared and can be purchased separately. The general titles of the five volumes are as follows: The higher institutions of learning in northern Germany, The higher institutions of learning in middle and southern Germany, Special questions relating to higher mathematical education, Mathematics in the technical schools, and Mathematics in the public schools and in the normal schools. Felix Klein of Göttingen is general editor of the series. M.

On April 5th of the present year the Spanish Mathematical Society was organized in Madrid, Spain. During the following month this Society began the publication of a monthly periodicel called *Revista de la Sociedad Matematica Espanola.* The first two numbers of this periodical contain 40 and 36 pages, respectively, and the subject matter of these numbers inspires considerable confidence in the success of the journal. It is divided into seven sections bearing the following headings: Biography, articles, bibliography, news, vocabulary, intermedium, and problems. American mathematicians should take an especial interest in this new Society and its journal in view of the extensive use of the Spanish language on this continent and in our recently acquired possessions. The rapid recent advances of Italy towards mathematical eminence may inspire us with the hope that the Spanish Mathematical Society may be the beginning of a new mathematical era among those who use the Spanish language. M.

Two matters of interest to mathematical readers received special attention at the meeting of the National Education Association in San Francisco in July. One was the presentation of the modified schemes of admission recently promulgated at Harvard University and at the Univerity of Chicago. The Harvard program was presented by Assistant Professor Davis of the department of physics, who gave in exceedingly clear manner a

most illuminating exposition of Harvard's practice heretofore with reference to the admission of students, and of the proposed system which goes into operation for the first time this fall. In a word, the new plan is a well considered combination of certificates from the schools and examinations at the University. It is proposed to accept the statement from the schools as to the ground covered in the various subjects by the candidate and his general character and standing as a student during his preparatory course; and no longer to set examinations for the purpose of determining his *quantitative* attainment in all the varied ramifications of his four years' work. Examinations, however, will still be given, but of a different character from those of the past. They will be set in four different subjects, three of which are specified by the University and one is to be selected by the candidate. These will be given all at one time and only when the candidate is ready for admission. The purpose of the examination is entirely *qualitative*, namely, to ascertain the candidate's ability to think and to express himself along the line of his preparatory study, and with this in view he is allowed to choose his own subject for one of the tests, presumably the one which is his favorite and in which he can best display his real strength. These papers are not to be graded by any percentage system and there will be no *passing* in one or more of the *separate* subjects. The candidate will either be admitted *without conditions* on the strength of his general ability as shown in these examination papers and by his credentials from the school, or he will be rejected and debarred from further opportunity for trial at that time. This plan is without doubt a great improvement over the old one and will aid greatly both in bringing Harvard into closer touch with the public high schools from which source she has hitherto drawn comparatively few of her students, and also in confounding the host of mere coaching schools which have long prospered as expert crammers for Harvard's special type of quantitative examinations.

The new Chicago program was supported by Associate Professors O. W. Caldwell of the department of biology in the School of Education, and H. E. Slaught of the department of mathematics, both of whom were members of a large committee of the faculty who have had the matter under consideration for the past two years. As is well known, the University of Chicago admits students to its collegiate departments on certificates from accredited schools. Hence the question was not one of examinations in one form or another set by the University, but it was rather a reconsideration of the relations of the University to the schools and the development of a better scheme of *coöporations*. To this end a committee of five representatives of the University and five representatives of the secondary schools in the Middle West have been holding numerous all day sessions during the past year and the results of their work were continually available for the committee of the faculty above mentioned. The chief considerations embodied in the new programs adopted by the faculty last June may be sum-

marized as follows, and it will be seen that these points refer not only to the work of the preparatory school but also to that of the University as related to the schools:

(1) *Itemized* specifications of the preparatory work are to be replaced by *group* specifications which will allow a wide range of selection but will demand *continuity* of work along two or three lines chosen by the pupil. Of the fifteen units required for admission a student from an accredited four years high school must present three units of English (the only specifically named subject), together with a major sequence of three units from one of groups: Ancient languages, modern languages, history, mathematics, science; a minor sequence of two units from another one of these groups; and finally two units chosen at will from these groups, making ten units in all from the standard academic subjects. The remainining five units of the fifteen he may present from any subjects which the accredited school accepts toward its graduation diploma.

(2) The *group continuity* thus demanded in the preparatory school must then be continued in the first year in college, as follows: (a) The student must continue the study of English as one of his freshman subjects. (2) He must also continue his major or minor sequence or in special cases a subject which he studied during his last year in the preparatory course.

(3) *Breadth of view* is assured by requiring the equivalent of two units in each of the groups: Modern languages, historical subjects (including economics and sociology), mathematics, sciences, either in the preparatory school or during the first two years in college.

(4) *Depth of attainment* is assured by requiring during the four years in college one sequence of nine majors in some chosen department or in related departments, and one sequence of six majors in some other department. (A major is a course four or five hours per week for twelve weeks.)

(5) It is further understood that the future accrediting of schools is to be based upon a carefully prepared system of comparative records running through both secondary school and college and that the mutual relationship and mutual responsibility of both school and college teachers are to be fostered and developed by a scheme of mutual visitation whereby the secondary teacher is to see how the pupil is handled in college and the college teacher is to come into closer touch with the problem as it presents itself to the schools.

It is believed that the scheme of corporation between the secondary school and the University will greatly aid in bridging the unwarranted and artificial chasm which has heretofore existed between them, will assist the student in his first year in college to avoid many of the fatal pit-falls which have beset him, and will enable the schools to devote themselves to the task of serving their communities most advantageously rather than to the patchwork business of meeting the arbitrary details of minutely specified college entrance requirements, since by this program any student who has success-

fully completed a four years' high school course and has devoted himself within reasonable limitations to *serious and prolonged study* of English and two other academic subjects may be admitted to college without conditions and may there find opportunity to continue his work for a college degree.

The question of special interest to teachers of mathematics is the fact that, according to the Chicago scheme, mathematics is made coordinate with modern an ancient languages, history and the sciences as *one* of five groups from which the two sequences must be chosen for college admission.

The other items of interest to teachers of mathematics is the preliminary report of the National Committee on Geometry which was presented at the meeting of the sections by the chairman, H. E. Slaught, of Chicago. This report is a pamphlet of about 80 pages reprinted from *School Science and Mathematics*, where it appeared in three installments in April, May, and June. The printed copies being in the hands of the audience, the whole time of the session was devoted to an exposition of the salient points by the chairman and to formal and informal discussion and criticism. As was to be expected a very large majority of the nearly two hundred present were teachers from the Pacific Coast, and the warmth and interest manifested in the discussions were characteristic of the free and untrammeled spirit of the West. A full report of the meeting will be printed in the proceedings of the Association, including the historical introduction of Professor Cojari. The remaining portions of the report will be carefully gone over by the committee during the present year in the light of all the suggestions and criticisms received and will be presented as a final report at the meeting of the Association next summer.

It may be said that very few unfavorable criticisms have been received from any source, and these are of minor importance, while the favorable comments are many and strong. The two points most ardently commended are (1) the attempt to distinguish the theorem of geometry on the basis of emphasis and to enable the teacher to get a view point of perspective in the subject, and (2), the attempt to improve the point of view with respect to problems and exercises.

An appropriation was voted by the finance committee of the Association for printing a large edition of the reports as soon as the committees shall be able to put the work into final form, at which time announcement will be made as to the terms on which it may be obtained by all who wish to receive it.

Professor A. L. McCarty is now at the head of the Department of Mathematics in the State Normal School, Cape Girardeau, Mo. F.

Professor Alois F. Kovarik has just returned to his work in the Department of Physics in the University of Minnesota, after a two years' leave of absence in research study in radio-activity with Professor Rutherford in England. F.

Professor B. F. Yanney, formerly of Mount Union College, is now at the head of the Department of Mathematics in Wooster University, Wooster, Ohio. F.

## BOOKS.

*Elements of the Differential and Integral Calculus.* By Wm. Anthony Granville, Ph. D., President of Pennsylvania College, with the Editorial Co-operation of Percy F. Smith, Ph. D., Professor of Mathematics in the Sheffield Scientific School of Yale University. Revised Edition. 8vo. Cloth. xv+463 pages. Price $2.50. Boston and Chicago: Ginn & Co.

This book has been very favorably received by teachers of the Calculus. Portraits of the two discoverers of the Calculus are inserted in the present edition. F.

*Physical Measurements.* By A. Wilmer Duff, Profossor of Physics in the Worcester Polytechnic Institute, and Arthur W. Ewell, Professor of Physics, in the Worcester Polytechnic Institute. Second Edition, Revised and Enlarged, with 78 Illustrations. 8vo. Cloth. x+258 pages. Price, $1.50. Philadelphia: P. Blakeston's Son & Co.

This manual contains a large variety of the most important experiments in the various subjects of Physics, such as Mechanics, Heat, Sound, Light, and Electricity and Magnetism. The experiments can be made with a minimum amount of apparatus. F.

*Exercises From Algebra for Secondary Schools.* By Charless Davison, Sc. D., Mathematical Master of King Edward's High School, Birmingham, Eng. 8vo. Cloth, vi+320 pp. Price with answers, $1.00. Cambridge, England: The University Press. G. P. Putnam's Sons, American Agents.

The book contains a fine collection of exercises and problems in the various subjects in elementary algebra from addition through the progressions, logarithms, permutation and combination, convergency and divergency of series, and exponential and logarithmic series. The answers are given at the end of the volume. The book will be found very useful to those teachers who wish to supplement the problems of the texts with well chosen problems from outside sources. F.

*Principles of Physics,* Designed for the Use as a Text Book of General Physics, by William Francis Magie, Ph. D., Henry Professor of Physics in Princeton University. 8vo. Cloth. ix+570 pages. Price, $3.00. New York: The Century Co.

This is a text book of rare excellence and one that will, we predict, be extensively used in our best colleges and universities. The book develops the subject in the historical order, thus putting in the clearest light those facts which are fundamental, showing the way theory grows out of facts, and leading the mind of the student along the pathway of discovery. The facts first discovered were those lying on the surface and thus most exposed. These are generally the easiest to understand. This is the reason why the historical order of study is pedagogically the best. Only a minimum of mathematics is required in reading this text. F.

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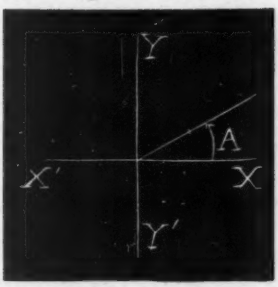
# REDUCTION OF THE TRIGONOMETRIC FUNCTIONS OF ANY ANGLE TO THE FUNCTIONS OF THE ANGLES IN A SMALL INTERVAL.

By DR. G. A. MILLER.

In view of the great practical importance of the reduction of the trigonometric functions of an angle to the similar functions of an angle in an interval which does not exceed 90°, it may be of interest to present a method of reduction which is simple and has some points of novelty even if this method is not decidedly superior to the one commonly employed in the textbooks on trigonometry. The method has, however, the strong point in its favor that it is applicable to many other problems and exhibits deep contact which does not appear so clearly in the methods commonly employed.

For simplicity of statement we shall always assume that the angle $A$ under consideration has its vertex at the origin of a rectangular system of coördinates and that the initial line coincides with the positive part of the $x$-axis. It is easy to see that we may find the geometric* angle which corresponds to the complement of $A$ by reflecting the terminal line of $A$ on the bisector of the first and third quadrants.




Similarly, we may find the supplement of $A$ by reflecting the terminal line of $A$ on the $y$-axis.

If we perform the operation of taking the complement ($c$) and of taking the supplement ($s$) of the finite angle $A$ and of the resulting angles, we obtain, in general, eight different geometric angles. We never obtain more than eight such angles, and we always obtain exactly eight except when the terminal line of $A$ lies either on one of the coördinate axes or on a bisector of one of the

* It is well known that each geometric angle has an infinite number of different measures. These measures will be called the analytic angles corresponding to the given geometric angle. Hence $\alpha$ and $\alpha+360°$ represent different analytic angles but they represent the same geometric angles. The trigonometric functions are based upon the geometric angles.